

Mathematics of the Galton Board: From Bernoulli Trials to the Gaussian Distribution

1. What is a Galton Board?

A **Galton Board** (also known as a **quincunx** or **bean machine**) is a physical device invented by Sir Francis Galton to demonstrate fundamental concepts in probability and statistics.

It consists of a vertical board with interleaved rows of pegs. When beads or shot are dropped from a funnel at the top, they bounce through the grid of pegs and collect into a series of vertical bins at the bottom. The resulting accumulation of beads forms a visible curve that illustrates how microscopic random events accumulate into macroscopic probabilistic patterns.

2. The Micro Level: What Each Peg Represents

At the individual peg level, the mechanics represent a simple binary random event:

- **A Single Peg = A Bernoulli Trial**
 - When a bead hits a single peg, it must bounce either to the left or to the right.
 - This individual interaction is modeled as a **Bernoulli trial**, where:
 - Bouncing Right = “Success” (with probability p)
 - Bouncing Left = “Failure” (with probability $q = 1 - p$)
 - In a standard, symmetric Galton Board, $p = 0.5$ and $q = 0.5$.

3. The Macro Level: What All the Pegs Represent

When a bead falls through n rows of pegs, it undergoes n independent Bernoulli trials in sequence:

- **The Entire Board = A Binomial Distribution $B(n, p)$**
 - The path a bead takes is defined by the total number of times it bounces right (k) out of n total rows.
 - The probability $P(k)$ that a bead lands in bin k is given by the Binomial distribution formula:

$$P(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

- **Mean (Center):** $\mu = n \cdot p$
- **Variance (Spread):** $\sigma^2 = n \cdot p \cdot (1 - p)$

4. Convergence to a Gaussian Distribution

As beads drop through the machine, the discrete Binomial distribution smoothes out into a continuous bell curve (Gaussian/Normal distribution).

The De Moivre–Laplace Theorem & CLT

This transition is governed by the **De Moivre–Laplace Theorem** (a special case of the **Central Limit Theorem**). It states that as the number of trials $n \rightarrow \infty$, the discrete Binomial distribution $B(n, p)$ converges toward a continuous Normal distribution $N(\mu, \sigma^2)$:

$$P(k) \approx \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{(k - np)^2}{2np(1-p)}\right)$$

As thousands of beads fall into the bins at the bottom, their collective heights trace out the Gaussian bell curve in real time.

5. Non-Symmetric Boards: What Happens When $p \neq 0.5$?

If the machine is modified—for example, by tilting the board or slanting the pegs so that $p = 0.7$ and $q = 0.3$ —it still converges to a Gaussian distribution.

Key Differences with $p \neq 0.5$:

1. **Shifted Center:** The peak moves to the side favored by p . Instead of being centered at the midpoint ($\mu = 0.5n$), the peak sits at $\mu = 0.7n$.
2. **Skewness at Small n :** With only a few rows of pegs, the distribution is visibly asymmetric (skewed to the right).

Why Skewness Disappears with Many Rows

The skewness (γ_1) of a Binomial distribution measures its asymmetry: